

COMPLEX HERMITE FUNCTIONS AS FOURIER-WIGNER TRANSFORM

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ABSTRACT. We prove that the complex Hermite polynomials $H_{m,n}$ on the complex plane \mathbb{C} can be realized as the Fourier-Wigner transform \mathcal{V} of the well-known real Hermite functions h_n on real line \mathbb{R} . This reduces considerably the Wong's proof [19, Chapter 21] giving the explicit expression of $\mathcal{V}(h_m, h_n)$ in terms of the Laguerre polynomials. Moreover, we derive a new generating function for the $H_{m,n}$ as well as some new integral identities.

1 INTRODUCTION

The so-called Fourier-Wigner transform is the windowed Fourier transform defined by

$$(1) \quad \mathcal{V}(f, g)(p, q) = \left(\frac{1}{2\pi} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle x + \frac{p}{2}, q \rangle} f(x + p) \overline{g(x)} dx$$

for every $(p, q) \in \mathbb{R}^d \times \mathbb{R}^d$ and every complex-valued functions $f, g \in L^2(\mathbb{R}^d)$. This transform is a basic tool to study the Weyl transform [5, 18, 19] and to interpret quantum mechanics as a form of nondeterministic statical dynamics [15]. It is also used to study the nonexisting joint probability distribution of positioned momentum in a given state [19]. In addition, the transform \mathcal{V} leads to the well-known Segal-Bargmann transform for a special window function [5, 17]. As basic property of this transform, one can use it to construct orthonormal bases for the Hilbert space $L^2(\mathbb{C}^d)$ from orthonormal bases of $L^2(\mathbb{R}^d)$.

In the present paper, we provide a new application of the Fourier-Wigner transform in the context of the complex Hermite polynomials $H_{m,n}$ [7, 10, 4]. More precisely, we realize $H_{m,n}(z; \bar{z})$ as the Fourier-Wigner transform of the well-known real Hermite functions h_n on \mathbb{R} . This reduces considerably the Wong's proof [19, Chapter 21] giving the explicit expression of $\mathcal{V}(h_n, h_m)$ in terms of the Laguerre polynomials. Moreover, we derive a new generating function for the complex Hermite polynomials $H_{m,n}$ as well as some new identities in the context of integral calculus.

The basic topics that we need in Fourier-Wigner transform, and in real and complex Hermite polynomials are collected in Section 2 and Section 3, respectively. In Section 4, we state and prove our main results. We end the paper with some concluding remarks.

2 THE FOURIER-WIGNER TRANSFORM

The Fourier-Wigner transform $\mathcal{V} : (f, g) \mapsto \mathcal{V}(f, g)$, given through (1), is a well defined bilinear mapping on $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, with

$$|\mathcal{V}(f, g)(p, q)| \leq \left(\frac{1}{2\pi} \right)^{\frac{d}{2}} \|f\|_{L^2(\mathbb{R}^d)} \|g\|_{L^2(\mathbb{R}^d)}$$

for every $(f, g) \in L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and $(p, q) \in \mathbb{R}^d \times \mathbb{R}^d$. It can be realized as a Fourier transform defined on \mathbb{R}^d by

$$\mathcal{F}(f)(\xi) := \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-i\langle y, \xi \rangle} f(y) dy.$$

In fact, we have $\mathcal{V}(f, g)(p, q) = \mathcal{F}(K_{f,g}(\cdot|p))(-q)$, where the function $y \mapsto K_{f,g}(y|p)$ belonging to $L^1(\mathbb{R}^d)$ is defined on \mathbb{R}^d by

$$(2) \quad K_{f,g}(y|p) = f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)}$$

for every given $f, g \in L^2(\mathbb{R}^d)$ and fixed $p \in \mathbb{R}^d$. More explicitly,

$$(3) \quad \mathcal{V}(f, g)(p, q) = \left(\frac{1}{2\pi}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{i\langle y, q \rangle} f\left(y + \frac{p}{2}\right) \overline{g\left(y - \frac{p}{2}\right)} dy.$$

An interesting result for \mathcal{V} is the Moyal's formula

$$\langle \mathcal{V}(f, g), \mathcal{V}(\varphi, \psi) \rangle_{L^2(\mathbb{C}^d)} = \langle f, \varphi \rangle_{L^2(\mathbb{R}^d)} \langle \psi, g \rangle_{L^2(\mathbb{R}^d)}; \quad f, g, \varphi, \psi \in L^2(\mathbb{R}^d),$$

giving rise to

$$\|\mathcal{V}(f, g)\|_{L^2(\mathbb{C}^d)}^2 = \|f\|_{L^2(\mathbb{R}^d)}^2 \|g\|_{L^2(\mathbb{R}^d)}^2.$$

Subsequently, we have $\mathcal{V}(L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)) \subset L^2(\mathbb{C}^d)$. Moreover, $\mathcal{V}(\mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)) \subset \mathcal{S}(\mathbb{C}^d)$ for every f and g in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

Using Moyal's formula and, it can be shown [19] that the Fourier-Wigner \mathcal{V} can be used to construct orthogonal bases of $L^2(\mathbb{C}^d)$ from those of $L^2(\mathbb{R}^d)$. More precisely, if $\{\varphi_k, k \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$, then $\{\varphi_{jk} = \mathcal{V}(\varphi_j, \varphi_k); j, k \in \mathbb{N}\}$ is an orthonormal basis of $L^2(\mathbb{C}^d)$ with

$$\|\varphi_{jk}\|_{L^2(\mathbb{C}^d)}^2 = \|\varphi_j\|_{L^2(\mathbb{R}^d)}^2 \|\varphi_k\|_{L^2(\mathbb{R}^d)}^2.$$

3 REAL AND COMPLEX HERMITE POLYNOMIALS

The classical real Hermite polynomials $H_n(x)$ are defined by the Rodrigues' formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}),$$

or also by their generating function

$$(4) \quad \sum_{m=0}^{+\infty} \frac{H_m(x)}{m!} t^m = e^{-t^2 + 2xt}.$$

Associated to H_n they are the Hermite functions

$$h_n(x) = e^{-\frac{x^2}{2}} H_n(x),$$

which constitute an orthogonal basis of the Hilbert space $L^2(\mathbb{R})$. An interesting result for the Hermite functions is the Mehler's formula [21, 9],

$$(5) \quad \sum_{m=0}^{+\infty} \frac{\lambda^m}{2^m m!} h_m(x) h_m(y) = g(x, y|\lambda)$$

fulfilled for $|\lambda| < 1$, where

$$(6) \quad g(x, y|\lambda) = \frac{1}{\sqrt{1-\lambda^2}} \exp\left(-\frac{1+\lambda^2}{2(1-\lambda^2)}(x^2 + y^2) + \frac{2\lambda}{1-\lambda^2}xy\right).$$

An extension of $H_m(x)$ to the complex variable are the so-called complex Hermite polynomials $H_{m,n}(z, \bar{z})$ introduced by Itô [13] in the context of complex Markov process. They can be defined by means of their generating function [7]

$$(7) \quad \sum_{m,n=0}^{+\infty} \frac{u^m}{m!} \frac{v^n}{n!} H_{m,n}(z, \bar{z}) = e^{-uv+zu+\bar{z}v}.$$

The explicit expression of $H_{m,n}$ in terms of the generalized Laguerre polynomials ([21]) $L_n^{(\alpha)}(x)$ is given by [12, 7]

$$(8) \quad H_{m,n}(z, \bar{z}) = (-1)^{\min(m,n)} (\min(m,n))! |z|^{|m-n|} e^{i(m-n)\arg(z)} L_{\min(m,n)}^{(|m-n|)}(|z|^2)$$

with $z = |z|e^{i\arg(z)}$. The polynomials $H_{m,n}$ constitute a complete orthogonal system of the Hilbert space $L^2(\mathbb{C}; e^{-|z|^2} d\lambda)$ and appear naturally when investigating the eigenvalue problem of some second order differential operators of Laplacian type [20, 14, 8].

Several interesting features of $H_{m,n}$ in connection with singular values of Cauchy transform [12], coherent states theory [2, 1], combinatory [11, 10] and signal processing [16, 3] have been studied recently. In the next section, we realize $H_{m,n}(z; \bar{z})$ as the Fourier-Wigner transform of the well-known real Hermite functions h_n on \mathbb{R} , and derive interesting identities of these polynomials.

4 MAIN RESULTS

In this section, we provide new applications of the Fourier-Wigner transform and see how it turns up in the context of complex Hermite polynomials. The first main result gives the explicit expression of $\mathcal{V}(h_m, h_n)$ in terms of the complex Hermite polynomials $H_{m,n}$. Namely, we assert

Theorem 4.1. *For every $p, q \in \mathbb{R}$, we have*

$$(9) \quad \mathcal{V}(h_m, h_n)(p, q) = (-1)^n \sqrt{2}^{m+n-1} e^{-\frac{p^2+q^2}{4}} H_{m,n}\left(\frac{p+iq}{\sqrt{2}}, \frac{p-iq}{\sqrt{2}}\right).$$

Proof. Note first that by making use of the generating function (4) for the real Hermite polynomials H_m , we get easily that

$$(10) \quad \sum_{m,n=0}^{+\infty} \frac{u^m}{m!} \frac{v^n}{n!} H_m\left(y + \frac{p}{2}\right) H_n\left(y - \frac{p}{2}\right) = e^{-(u^2+v^2)+2y(u+v)+p(u-v)}.$$

On the other hand, according to the expression of \mathcal{V} given by (3), we can write

$$\begin{aligned} \mathcal{V}(h_m, h_n)(p, q) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iyq} e^{-\frac{(y+\frac{p}{2})^2}{2}} e^{-\frac{(y-\frac{p}{2})^2}{2}} H_m\left(y + \frac{p}{2}\right) H_n\left(y - \frac{p}{2}\right) dy \\ &= \frac{e^{-\frac{p^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-y^2+iyq} H_m\left(y + \frac{p}{2}\right) H_n\left(y - \frac{p}{2}\right) dy. \end{aligned}$$

By means of (10), we obtain

$$\sum_{m,n=0}^{+\infty} \frac{u^m}{m!} \frac{v^n}{n!} \mathcal{V}(h_m, h_n)(p, q) = \frac{e^{-\frac{p^2}{4}}}{\sqrt{2\pi}} e^{-(u^2+v^2)+p(u-v)} \int_{-\infty}^{+\infty} e^{-y^2+(iq+2(u+v))y} dy.$$

More explicitly, we have

$$(11) \quad \sum_{m,n=0}^{+\infty} \frac{u^m}{m!} \frac{v^n}{n!} \mathcal{V}(h_m, h_n)(p, q) = \frac{e^{-\frac{p^2+q^2}{4}}}{\sqrt{2}} e^{2uv+(p+iq)u-(p-iq)v}$$

which follows by applying the well-known expression of the classical Gauss integral given by

$$(12) \quad \int_{\mathbb{R}} e^{-\alpha y^2 + \beta y} dy = \left(\frac{\pi}{\alpha}\right)^{1/2} e^{\beta^2/4\alpha}$$

with $\alpha > 0$ and $\beta \in \mathbb{C}$. Now, by setting $z = p + iq$, we get

$$(13) \quad \sum_{m,n=0}^{+\infty} \frac{u^m v^n}{m! n!} \mathcal{V}(h_m, h_n)(p, q) = \frac{e^{-\frac{|z|^2}{4}}}{\sqrt{2}} e^{2uv + zu - \bar{z}v}.$$

In the right-hand-side of the last equality, we recognize the generating function (7) of the complex Hermite polynomials $H_{m,n}$. Therefore,

$$\begin{aligned} \sum_{m,n=0}^{+\infty} \frac{u^m v^n}{m! n!} \mathcal{V}(h_m, h_n)(p, q) &= \frac{e^{-\frac{|z|^2}{4}}}{\sqrt{2}} \sum_{m,n=0}^{+\infty} \frac{(\sqrt{2}u)^m (-\sqrt{2}v)^n}{m! n!} H_{m,n} \left(\frac{z}{\sqrt{2}}, \frac{\bar{z}}{\sqrt{2}} \right) \\ &= \frac{e^{-\frac{|z|^2}{4}}}{\sqrt{2}} \sum_{m,n=0}^{+\infty} \frac{u^m v^n}{m! n!} \left[(-1)^n \sqrt{2}^{m+n} H_{m,n} \left(\frac{z}{\sqrt{2}}, \frac{\bar{z}}{\sqrt{2}} \right) \right]. \end{aligned}$$

By identifying the two power series, we obtain (9). ■

The special case of $p = 0$ in (9) yields the following

Corollary 4.2. *For every $t \in \mathbb{R}$, we have*

$$(14) \quad \int_{-\infty}^{+\infty} H_m(y) H_n(y) e^{-y^2 - ity} dy = (-1)^n \sqrt{\pi} \sqrt{2}^{m+n} e^{-\frac{t^2}{4}} H_{m,n} \left(-\frac{it}{\sqrt{2}}, \frac{it}{\sqrt{2}} \right).$$

While when specifying $q = 0$, we can deduce the following

Corollary 4.3. *For every $t \in \mathbb{R}$, we have*

$$(15) \quad \int_{-\infty}^{+\infty} H_m \left(y + \frac{t}{2} \right) H_n \left(y - \frac{t}{2} \right) e^{-y^2} dy = (-1)^n \sqrt{\pi} \sqrt{2}^{m+n} H_{m,n} \left(\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}} \right).$$

Remark 4.4. *By taking $t = 0$ in (14) and (15), we recover the well-known formula*

$$\int_{-\infty}^{+\infty} H_m(y) H_n(y) e^{-y^2} dy = \sqrt{\pi} 2^m m! \delta_{m,n}$$

since $H_{m,n}(0, 0) = (-1)^m m! \delta_{m,n}$.

As an immediate consequence of Theorem 4.1, combined with the fact that \mathcal{V} transforms orthogonal basis of $L^2(\mathbb{R})$ to an orthogonal basis of $L^2(\mathbb{C})$, we recover the following well-known result (see [12, 6, 7, 11, 10]).

Corollary 4.5. *The complex Hermite polynomials $H_{m,n}$ constitute an orthogonal basis of the Hilbert space $L^2(\mathbb{C}, e^{-|z|^2} dx dy)$; $z = x + iy$.*

In the sequel, we will investigate further consequences of Theorem 4.1. We begin, by establishing a new generating function for the complex Hermite polynomials $H_{m,n}$ with $m = n$. Namely, we assert

Theorem 4.6. *For every positive real number $0 < \lambda < 1$, we have*

$$(16) \quad \sum_{m=0}^{+\infty} \frac{\lambda^m}{m!} H_{m,m}(z, \bar{z}) = \frac{e^{\frac{\lambda}{1+\lambda}|z|^2}}{1+\lambda}.$$

Proof. Starting from (9), we can write

$$(17) \quad H_{m,m}(z, \bar{z}) = (-1)^m \frac{\sqrt{2}}{2^m} e^{\frac{|z|^2}{2}} \mathcal{V}(h_m, h_m)(\sqrt{2}p, \sqrt{2}q)$$

with $z = p + iq$; $p, q \in \mathbb{R}$. Therefore, we get

$$\begin{aligned} \sum_{m=0}^{+\infty} \frac{\lambda^m}{m!} H_{m,m}(z, \bar{z}) &= \sqrt{2} e^{\frac{|z|^2}{2}} \sum_{m=0}^{+\infty} \frac{(-\lambda)^m}{2^m m!} \mathcal{V}(h_m, h_m)(\sqrt{2}p, \sqrt{2}q) \\ &\stackrel{(3)}{=} \frac{e^{\frac{|z|^2}{2}}}{\sqrt{\pi}} \sum_{m=0}^{+\infty} \frac{(-\lambda)^m}{2^m m!} \int_{\mathbb{R}} e^{i\sqrt{2}qy} h_m\left(y + \frac{p}{\sqrt{2}}\right) h_m\left(y - \frac{p}{\sqrt{2}}\right) dy \\ &= \frac{e^{\frac{|z|^2}{2}}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{i\sqrt{2}qy} \left(\sum_{m=0}^{+\infty} \frac{(-\lambda)^m}{2^m m!} h_m\left(y + \frac{p}{\sqrt{2}}\right) h_m\left(y - \frac{p}{\sqrt{2}}\right) \right) dy. \end{aligned}$$

In the last equality we recognize the Mehler's formula (5) for the Hermite functions. Whence, it follows

$$(18) \quad \sum_{m=0}^{+\infty} \frac{\lambda^m}{m!} H_{m,m}(z, \bar{z}) = \frac{e^{\frac{|z|^2}{2}}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{i\sqrt{2}qy} g\left(y + \frac{p}{\sqrt{2}}, y - \frac{p}{\sqrt{2}} \middle| -\lambda\right) dy.$$

Now, since

$$g\left(y + \frac{p}{\sqrt{2}}, y - \frac{p}{\sqrt{2}} \middle| -\lambda\right) = \frac{1}{\sqrt{1-\lambda^2}} e^{-\frac{1+\lambda}{1-\lambda}y^2 - \frac{1-\lambda}{2(1+\lambda)}p^2}.$$

Equation (18) becomes

$$\sum_{m=0}^{+\infty} \frac{\lambda^m}{m!} H_{m,m}(z, \bar{z}) = \frac{e^{\frac{|z|^2}{2} - \frac{1-\lambda}{1+\lambda}\frac{p^2}{2}}}{\sqrt{\pi}\sqrt{1-\lambda^2}} \int_{\mathbb{R}} e^{-\frac{1+\lambda}{1-\lambda}y^2 + i\sqrt{2}qy} dy.$$

Making appeal to (12) with $\alpha = \frac{1+\lambda}{1-\lambda} > 0$, for $0 < \lambda < 1$, and $\beta = i\sqrt{2}q$, we get (16). ■

5 CONCLUDING REMARKS

Instead of h_n , we consider the orthonormal basis of $L^2(\mathbb{C})$ given by

$$e_n(x) = \frac{h_n(x)}{\|h_n\|} = \frac{h_n(x)}{\sqrt{2^n n! \sqrt{\pi}}}.$$

Thus according to (8), giving the expression of the complex Hermite polynomials in terms of the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$, we can rewrite the result of Theorem 4.1 as

$$\mathcal{V}(e_m, e_n)(p, q) = \frac{1}{\sqrt{2\pi}} e^{-\frac{|z|^2}{4}} \begin{cases} \frac{\sqrt{n!}}{\sqrt{m!}} \frac{z^{m-n}}{\sqrt{2}^{m-n}} L_n^{(m-n)}\left(\frac{|z|^2}{2}\right) & \text{if } m \geq n \\ (-1)^{n+m} \frac{\sqrt{m!}}{\sqrt{n!}} \frac{\bar{z}^{n-m}}{\sqrt{2}^{n-m}} L_m^{(n-m)}\left(\frac{|z|^2}{2}\right) & \text{if } n \geq m \end{cases},$$

where $p + iq = z$. This reads equivalently as

$$(19) \quad \mathcal{V}(e_{j+k}, e_j)(p, q) = \left(\frac{j!}{2\pi 2^k (j+k)!} \right)^{1/2} z^k L_j^{(k)}\left(\frac{|z|^2}{2}\right) e^{-\frac{|z|^2}{4}}$$

$$(20) \quad \mathcal{V}(e_j, e_{j+k})(p, q) = (-1)^k \left(\frac{j!}{2\pi 2^k (j+k)!} \right)^{1/2} \bar{z}^k L_j^{(k)}\left(\frac{|z|^2}{2}\right) e^{-\frac{|z|^2}{4}}.$$

Whence, we recover the result established by Wong in [19] giving the explicit expression of $\mathcal{V}(e_n, e_m)$ in terms of the Laguerre polynomials ($i\bar{z}$ in Wong's notation is z in ours). However, our proof reduces considerably the one given by Wong.

We conclude this paper by noting that by adopting the same approach as above, one can introduce a new class of orthogonal polynomials on the quaternion $\mathbb{R}^4 = \mathbb{H}$. This is the subject of another paper.

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